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# BPS Monopoles and Electromagnetic Duality<sup>1</sup>

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## ABSTRACT

We review our recent work on the BPS magnetic monopoles and its relation to the electromagnetic duality in the  $N = 4$  supersymmetric Yang-Mills systems with an arbitrary gauge group. The gauge group can be maximally or partially broken. The low energy dynamics of the massive and massless magnetic monopoles are approximated by the moduli space metric. We emphasize the possible connection between the nature of the monopole moduli space with unbroken gauge group and the physics of mesons and baryons in QCD.

## 1 Introduction

Recently there has been a considerable progress in our understanding of the electromagnetic duality. This duality is a duality between strong and weakly interacting theories and so intrinsically nonperturbative. Especially a simple case appears when we consider the  $N = 4$  supersymmetric Yang-Mills systems which can be regarded self-dual under the electromagnetic duality. (For the recent detailed review, see Ref.[1].) Thus, the spectrum of magnetic monopoles should match to that of electrically charged particles exactly.

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In this talk, I will describe some recent works[2, 3, 4] done in collaboration with Erick Weinberg and Piljin Yi, concerning the low energy dynamics of magnetic monopoles in the systems with larger gauge group than  $SU(2)$ . The gauge symmetry can be maximally broken to abelian subgroups or partially broken with unbroken nonabelian gauge group.

The outline of this talk is as follows: In Sec.2, we briefly review the magnetic monopoles and the electromagnetic duality. In Sec.3, we consider the BPS magnetic monopole configurations. We discuss zero modes of the BPS configurations and show that the duality of the electric and magnetic sectors entails the better understanding of the low energy dynamics of the BPS monopoles. In Sec. 4, we discuss how the low energy dynamics of the BPS monopoles are approximated by the moduli space dynamics, or the zero mode dynamics. I briefly describe how to derive the moduli space metric for the distinct fundamental monopoles. In Sec. 5, two specific examples  $SU(3) \rightarrow U(1)^2$  and  $SU(4) \rightarrow U(1)^2 \times SU(2)$  are studied. In Sec. 6, we close with some concluding remarks.

## 2 Duality

The duality between electricity and magnetism has been fascinating us quite a while. Initially, it has originated from the invariance of the free Maxwell equations under the global phase rotation  $(\mathbf{E} + i\mathbf{B}) \rightarrow e^{i\alpha} (\mathbf{E} + i\mathbf{B})$ . When the electric and magnetic currents,  $j_e^\mu$  and  $j_m^\mu$ , are introduced, the Maxwell equation becomes

$$\nabla \cdot (\mathbf{E} + i\mathbf{B}) = j_e^0 + ij_m^0, \quad (1)$$

$$\partial_t(\mathbf{E} + i\mathbf{B}) - i\nabla(\mathbf{E} + i\mathbf{B}) = \mathbf{j}_e + i\mathbf{j}_m. \quad (2)$$

They are still invariant if we also rotate the four current  $j_e^\mu + ij_m^\mu \rightarrow e^{i\alpha}(j_e^\mu + ij_m^\mu)$ . This duality allows us to understand the classical interaction between point particles carrying electric and magnetic charges.

Dirac[5] showed that quantum mechanical interaction between electrically charged particles and magnetic monopoles can be consistently implemented only if the quantization law,

$$qg = 2\pi n\hbar \quad (3)$$

with an integer  $n$ , between electric charge  $q$  and magnetic charge  $g$  is satisfied. This quantization is also consistent with the angular momentum quantization. Not only this quantization law explains the electric charge quantization in a presence of a single monopole anywhere in the universe, it is also invariant under a more restricted ‘electromagnetic’ duality,  $(E, B) \rightarrow (B, -E)$  and  $(j_e^\mu, j_m^\mu) \rightarrow (j_m^\mu, -j_e^\mu)$ . In terms of the electric charge unit  $e$ , the minimum magnetic charge should be  $2\pi\hbar/e$ . Thus when the electromagnetic coupling constant is small, the magnetic coupling constant is large, and vice versa.

By considering two interacting dyons of charge  $(q_i, g_i)$  with  $i = 1, 2$ , Schwinger and Zwanziger[6] extended the Dirac’s law to

$$q_1 g_2 - q_2 g_1 = 2\pi\hbar. \quad (4)$$

By considering the effect of the  $CP$  violating  $\theta$  term in the Maxwell systems and its parent Yang-Mills-Higgs systems, Witten[7] showed that the above quantization law can be implemented by pure electrically charged particles of charge in unit of  $e$  and dyons of quantized magnetic charge  $g = (2\pi/e)n_m$  and fractional electric charge

$$q = e(n_e + \frac{\theta}{2\pi}n_m), \quad (5)$$

where  $n_e, n_m$  are integers.

On the other hand, t’Hooft and Polyakov[8] found magnetic monopoles as solitons in Yang-Mills Higgs systems where the gauge group  $SU(2)$  is broken to  $U(1)$  by the Higgs mechanism. The magnetic charge is topologically quantized  $g = 4\pi/e$  where  $e$  is the charge of elementary charged vector bosons. When the potential of the Higgs field vanishes, Bogomolny and others[9] found a bound on the energy by electric and magnetic charges  $(q, g)$ ,

$$E \geq v\sqrt{g^2 + q^2}, \quad (6)$$

which is saturated by every elementary particle and dyons. In this system, there exist also elementary massive charged vector bosons  $W_\mu^\pm$  whose mass saturate the above bound.

Montonen and Olive[10] proposed that in this theory holds the electromagnetic duality which transforms  $(e, g) \rightarrow (g, -e)$  and  $(n_e, n_m) \rightarrow (n_m, -n_e)$ . Soon it was realized by Osborn[11] that to match the spectrum of the electric

sector to that of the magnetic sector in spin content, one needs the  $N = 4$  supersymmetry. In this case both electric and magnetic sectors saturate the Bogomolny bound and so belong to the short representation of the supersymmetry with the maximal spin one[12].

The  $N = 4$  supersymmetric theory is finite and there is no running coupling constant and so the classical bound is expected to remain exact quantum mechanically. When the initial coupling constant is small, elementary particles are interacting weakly and magnetic monopoles are strongly interacting. As the size  $\sim 1/(ev)$  and mass  $\sim v/e$  of magnetic monopoles are much larger than those of elementary particles and we can approach the monopole physics semiclassically.

Instead if we make the coupling constant to be big, the theory becomes a strongly interacting gauge theory. In this case elementary particles are interacting strongly and solitons are interacting weakly. The mass of monopoles becomes smaller than that of elementary particles and the size of monopoles becomes smaller than the Compton length of magnetic monopoles. Thus, we cannot treat magnetic monopoles as classical solitons and so seem to be lost.

However if the electromagnetic duality holds, there would be a weakly interacting dual gauge theory where magnetic monopoles appear as elementary particles and  $W$  bosons appear as solitons. As the dual theory itself is the  $N = 4$  supersymmetric Yang-Mills theory, the dual theory is identical to the original theory with the coupling constant  $4\pi/e$ .

Thus the dynamics of strongly interacting  $W$  bosons in the theory with coupling constant  $4\pi/e$  with small  $e$  is identical to that of strongly interacting magnetic monopoles in the theory with coupling constant  $e$ . Thus the understanding of strongly interacting magnetic monopoles implies an understanding of strongly interacting  $W$  bosons.

There are two main questions to be addressed in this context. First question is about the validity of the duality. We do not know any rigorous dual transformation of the theory. However we can still test the duality. If the duality holds, the spectrum of electrically charged particles should match that of magnetic monopoles. It would be interesting to find other nontrivial tests of the duality. Second question is about the implications of the duality. As I argued before, the understanding of the magnetic monopole dynamics implies that of strongly interacting elementary particles.

The electromagnetic duality can be generalized into several directions. When the  $\theta$  parameter is introduced, the electromagnetic duality can be

generalized to the  $S$ -duality. In this context, Sen[13] provided the evidence for the  $S$ -duality, by finding the bound states of two identical magnetic monopoles with odd number of electric charge. Another direction is to consider the more general gauge group than  $SU(2)$ . With a larger gauge group one can consider the case where the unbroken gauge group is not purely abelian. When the duality is generalized to the unbroken nonabelian gauge group[14], its meaning becomes more subtle as we will see later.

### 3 BPS Monopoles

As emphasized before, the magnetic monopole configurations in the  $N = 4$  supersymmetric Yang-Mills theory are described by the BPS magnetic monopoles. Since there are six independent scalar fields in the system, the magnetic monopole dynamics is rather different at the generic points of the symmetry breaking. The monopole dynamics is rich when the vacuum expectation values of the six scalar fields are parallel and this is the case we will consider here. For more general case, see a recent article[15].

The energy of a general field configuration is bounded by the topological quantity which is a function of its electric and magnetic charge. The field configurations which saturate this BPS bound satisfy the first order BPS equation.

We start with an arbitrary gauge group  $G$  of rank  $r$ . Its generators are made of  $r$  commuting operators  $H^s$  and raising and lowering operators  $E_{\pm\alpha}$ . We choose the vacuum expectation value of the Higgs field as  $\Phi_0 = \mathbf{h} \cdot \mathbf{H}$  in the Cartan subalgebra. When the gauge symmetry is maximally broken to  $U(1)^r$ , there is a unique set of simple roots  $\beta_a$  such that  $\beta_a \cdot \mathbf{h} > 0$ . When the gauge symmetry is partially broken to  $K \times U(1)^{r-k}$  with a semisimple group  $K$  of rank  $k$ , there exists a unique set of simple roots  $\beta_a, \gamma_i$  modulo the Weyl group of  $K$ , where  $\gamma_i$  are the simple roots of  $K$  and so  $\gamma_i \cdot \mathbf{h} = 0$ .

In the direction chosen to define  $\Phi_0$ , the asymptotic magnetic field of *BPS* configuration will be of form

$$\vec{B} = \frac{\hat{r}}{4\pi r^2} \mathbf{g} \cdot \mathbf{H}, \quad (7)$$

whose magnetic charge  $\mathbf{g}$  satisfy the topological quantization condition [16]

$$\mathbf{g} = \frac{4\pi}{e} \left[ \sum_{a=1}^{r-k} n_a \boldsymbol{\beta}_a^* + \sum_{j=1}^k q_j \boldsymbol{\gamma}_j^* \right], \quad (8)$$

where  $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}/\boldsymbol{\alpha}^2$  is the dual of the root  $\boldsymbol{\alpha}$  and the  $n_a$  and  $q_j$  are non-negative integers. In the gauge group  $SU(N)$ , we choose the normalization  $\alpha^* = \alpha$ . The  $n_a$ 's are the topologically conserved charges. For a given solution they are uniquely determined and gauge invariant, even though the corresponding  $\boldsymbol{\gamma}_a$  may not be. The  $q_j$  are neither gauge invariant nor conserved except their sum.

For maximal symmetry breaking to  $U(1)^r$ , there is a unique fundamental monopole solution associated with each of the  $r$  topological charges. To obtain these, we first note that any root  $\boldsymbol{\alpha}$  defines an  $SU(2)$  subgroup generated by the raising and lowering operators  $\mathbf{E}_{\pm}\boldsymbol{\alpha}$ . One can embed the  $SU(2)$  magnetic monopole of unit charge to get a spherically symmetric monopole solution for given root  $\boldsymbol{\alpha}$ . For each simple root  $\boldsymbol{\beta}_a$ , the monopole has the unit topological magnetic charge  $\mathbf{g} = (4\pi/e)\boldsymbol{\beta}_a^*$  and mass  $m_a = (4\pi/e)\mathbf{h} \cdot \boldsymbol{\beta}_a^*$ . All other BPS solutions can be understood as multimonopole solutions containing  $N = \sum_{a=1}^r n_a$  fundamental monopoles[17]. Especially the rotationally invariant monopole solutions for the composite positive roots  $\boldsymbol{\alpha}$  is not fundamental. The moduli space for these multimonopole solutions has  $4N$  dimensions, corresponding to three position variables and a single  $U(1)$  phase for each of the component fundamental monopoles[17].

Matters are somewhat more complicated when the unbroken gauge group is nonabelian [18]. If the long-range magnetic field has a nonabelian component (i.e., if  $\mathbf{g} \cdot \boldsymbol{\gamma}_j \neq 0$ , the index theory methods used to count zero modes in Refs. [17] and [18] fail for technical reasons related to the slow falloff of the nonabelian field at large distance. If the total magnetic charge does not have the nonabelian component so that  $\mathbf{g} \cdot \boldsymbol{\gamma}_j = 0$ , these difficulties do not arise and the number of normalizable zero modes is

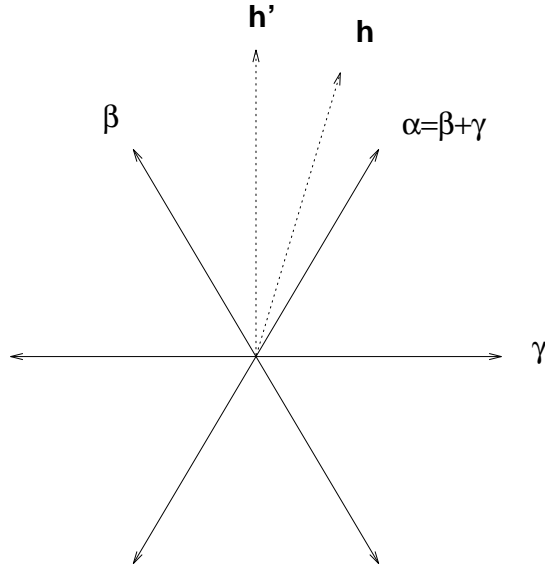
$$N = 4 \left[ \sum_a n_a + \sum_j q_j \right]. \quad (9)$$

Let us illustrate the above ideas in the case where the  $SU(3)$  gauge group is broken to either  $U(1)^2$  or  $SU(2) \times U(1)$ . In the maximally broken case,

$\Phi_0 = \mathbf{h} \cdot \mathbf{H}$  and two simple positive roots are  $\beta, \gamma$  as shown in Fig. 1. There are fundamental monopoles corresponding to simple dual roots  $\beta^*$  and  $\gamma^*$ , but there is only a composite monopole for root  $\alpha$ . As  $\mathbf{h} \rightarrow \mathbf{h}'$ , the unbroken gauge group becomes  $SU(2) \times U(1)$  as  $\mathbf{h}' \cdot \gamma = 0$ . The unbroken  $SU(2)$  is associated with the raising and lowering operators  $\mathbf{E}_{\pm\gamma}$ . The minimum magnetic charge without the nonabelian component is

$$\mathbf{g} = \frac{4\pi}{e}(2\beta^* + \gamma^*) \quad (10)$$

because  $\mathbf{g} \cdot \gamma = 0$ , which are composed of two massive and one massless monopoles. The number of zero modes for the above magnetic charge is 12.



**Figure 1:** Root diagram for  $SU(3)$ . When  $\Phi_0 = \mathbf{h} \cdot \mathbf{H}$ , the symmetry is maximally broken. When  $\Phi_0 = \mathbf{h}' \cdot \mathbf{H}$ , the symmetry is partially broken.

In the maximally broken symmetry phase there are as many charged vector bosons as the number of positive roots, whose number is much larger

than that of simple roots if the gauge group is bigger than  $SU(2)$ . On the other hand we just argued that classically the number of fundamental monopoles is identical to that of simple roots. In the above  $SU(3)$  example there are fundamental monopoles corresponding to the simple roots  $\beta^*, \gamma^*$ , but not for the composite root  $\alpha^*$ .

This seems to contradict with the duality hypothesis that the spectrum of charged particles should match exactly that of magnetic monopoles. However the duality is an intrinsically quantum mechanical statement: one has to study quantum mechanical spectrum of magnetic monopoles, which raises a possible existence of the quantum mechanical bound state of fundamental monopoles for each composite root. As the BPS mass formula is expected to be exact even in quantum mechanically in the  $N = 4$  supersymmetric theory, the bound energy of these composite monopoles would be zero. It seems hard to find such threshold bound states of magnetic monopoles directly in the full quantum field theory. However, one can approximate the very low energy dynamics of magnetic monopoles by the nonrelativistic moduli space dynamics, whose quantum mechanics may allow such expected bound states. Hence, let us now turn our attention to the magnetic monopole moduli space.

## 4 Moduli Space Approximation

The BPS configurations for a given magnetic charge, and so the same energy, are parameterized by the  $N$  collective coordinates  $z_\alpha, \alpha = 1 \dots N$ , or moduli up to local gauge transformations. The space of gauge-inequivalent BPS configurations for a given magnetic charge,

$$A_\mu(\mathbf{x}; z_\alpha) = (\mathbf{A}(\mathbf{x}; z_\alpha), A_4 = \Phi(\mathbf{x}, z_\alpha)), \quad (11)$$

is called the moduli space of solutions. The  $N$  zero modes  $\delta_\alpha A_\mu = \partial A_\mu / \partial z_\alpha - D_\mu \epsilon_\alpha$  satisfy the linearized BPS equation and are chosen to satisfy the background gauge

$$D_\mu \delta_\alpha A_\mu = 0 \quad (12)$$

with  $\partial_4 = 0$ . When we consider the fluctuations around the BPS magnetic monopole configurations, there are massless modes and massive modes. If the initial energy is arbitrary small, the dynamics of BPS monopoles may be approximated by that of moduli[19]. The initial field configuration at a given



time will be characterized by  $A_\mu(\mathbf{x}, z_\alpha(t))$  and its time derivative,  $\dot{z}_\alpha \delta_\alpha A_\mu$  in the  $A_0 = 0$  gauge. The Gauss law constraint on the initial configuration is exactly the background gauge (12).

Since there is no force between monopoles at rest, one expect the low energy dynamics is given by the kinetic part of the Yang-Mills-Higgs Lagrangian. In the  $A_0 = 0$  gauge, this becomes

$$L = \frac{1}{2} G_{\alpha\beta}(z_\alpha) \dot{z}_\alpha \dot{z}_\beta \quad (13)$$

where  $G_{\alpha\beta}(z_\alpha) = \int d^3x \operatorname{tr} \delta_\alpha A_\mu \delta_\beta A_\mu$ . While one can study some characteristics of this metric, it is hard to obtain directly from the BPS field configurations which themselves are not known in general. However some formal characteristics of the metric can be deduced from this. The important property of the metric is that it is hyperkähler. This property is also related to the field theoretical supersymmetry which should be incorporated into the Lagrangian (13) to be consistent [20].

The full moduli space and its metric are known for two identical monopoles[21, 2, 22, 23]. For  $N > 2$  the metric for the case where all the component fundamental monopoles are all distinct was given in Ref. [3]; for all other cases the explicit form of the metric is known only for the region of moduli space corresponding to widely separated fundamental monopoles[25, 3].

There are several approaches to calculate the moduli space metric. Here we focus on the approach taken by Manton and Gibbons[24, 25]. The metric determines the interaction between BPS magnetic monopoles of low kinetic energy and vice versa. Once we understand the interaction between magnetic monopoles, we can deduce the metric. The interaction between  $n$  fundamental BPS magnetic monopoles becomes particularly simple when their mutual distances are very large. In this large separation, the electric charge of each monopole is conserved as the possible violating term is exponentially small. Thus, it is easier to consider the interaction between  $n$  fundamental dyons in large separation. The interaction between the dyons in large separation becomes purely electromagnetic and scalar in nature. Once the nonrelativistic Lagrangian for dyons is obtained, the Lagrangian for monopoles can be obtained by the Legendre transformation of the electric charge to the phase variables.

Specifically we consider the  $r$  distinct fundamental monopoles in the maximally broken gauge group  $G$ . The  $a$ -th monopole associated with the simple

root  $\beta_a$  has the position  $\mathbf{x}_a$  and the phase variable  $\xi_a$ . The metric obtained from the method mentioned previously is

$$\mathcal{G} = \frac{1}{2} M_{ab} d\mathbf{x}_a \cdot d\mathbf{x}_b + \frac{g^4}{2(4\pi)^2} (M^{-1})_{ab} (d\xi_a + \mathbf{W}_{ac} \cdot d\mathbf{x}_c) (d\xi_b + \mathbf{W}_{bd} \cdot d\mathbf{x}_d), \quad (14)$$

where

$$\begin{aligned} M_{aa} &= m_a - \sum_{c \neq a} \frac{g^2 \beta_a^* \cdot \beta_c^*}{4\pi r_{ac}}, \\ M_{ab} &= \frac{g^2 \beta_a^* \cdot \beta_b^*}{4\pi r_{ab}} \quad \text{if } a \neq b, \end{aligned} \quad (15)$$

with  $m_a = g \beta_a^* \cdot \mathbf{h}$ , and

$$\begin{aligned} \mathbf{W}_{aa} &= - \sum_{c \neq a} \beta_a^* \cdot \beta_c^* \mathbf{w}_{ac}, \\ \mathbf{W}_{ab} &= \beta_a^* \cdot \beta_b^* \mathbf{w}_{ab} \quad \text{if } a \neq b. \end{aligned} \quad (16)$$

with  $\mathbf{w}_{ab} = \mathbf{w}(\mathbf{x}_a - \mathbf{x}_b)$  being value at  $\mathbf{x}_a$  of the Dirac potential due to the  $b$ -th monopole. The  $q$ -independent part of the monopole rest energies has been omitted. The fact that this asymptotic metric is hyperkähler can be shown trivially, following the argument by Gibbons and Manton[25]. The key ingredient is that  $\nabla 1/r = \nabla \times \mathbf{w}(r)$ . The question is whether the asymptotic metric when it is extended in the interior region is nonsingular.

Neither of these objections arises for the moduli space corresponding to a collection of several distinct fundamental monopoles in a larger group, provided that each corresponds to a different simple root. In this case, one can show that the metric is nonsingular everywhere by going to the center of mass frame[3]. The metric has the right isometry: the rotational symmetry and the  $U(1)$  symmetry for each conserved  $U(1)$  charge. More recently it has been strongly argued that the metric in this case is indeed exact[26].

## 5 Examples of the Moduli Space Metric

Here we discuss in detail two simple examples whose moduli space is known. First we discuss the case where the gauge symmetry is maximally broken to the abelian subgroups. Then we discuss the case where the symmetry is partially broken with unbroken nonabelian group.

## 5.1 $SU(3) \rightarrow U(1)^2$

For two distinct fundamental monopoles  $M_{\beta}$  and  $M_{\gamma}$  with the Dynkin diagram shown in Figure 1, the moduli space metric has been obtained from Eq.(14). In terms of the center of mass coordinates  $\mathbf{R}, \chi$  and the relative coordinates  $\mathbf{r}, \psi$ , the geometry of the center of mass coordinates is shown to be  $R^4$  and that of the relative coordinate[22, 23, 2] is the Taub-NUT space with the metric

$$\mathcal{G}_{\text{rel}}^{(2)} = \mu \left( \left(1 + \frac{2l}{r}\right) d\mathbf{r}^2 + 4l^2 \left(1 + \frac{2l}{r}\right)^{-1} (d\psi + \mathbf{w}(\mathbf{r}) \cdot d\mathbf{r})^2 \right). \quad (17)$$

The Taub-NUT metric is smooth everywhere including the origin if the  $\psi$  has a period of  $4\pi$ , which is true as one can see from the charge quantization.

In addition, there are identification maps forming the integer group  $Z$  on the cylinder  $(\chi, \psi)$ . The result is that the total moduli space is of the form [2]

$$\mathcal{M} = R^3 \times \frac{R^1 \times \mathcal{M}_0}{Z}, \quad (18)$$

where  $\mathcal{M}_0$  is the Taub-NUT manifold.

The relative metric possesses the  $SU(2)$  rotational symmetry and the global  $U(1)$  symmetry, which are required from the physics of two distinct dyons. Our metric is obtained from the interaction between monopoles in large separation. However there is a complete classification of four dimensional hyperkähler spaces in four dimensions with the rotational symmetry acting on three dimensional space[21]. Among them only one whose symmetry and asymptotic form match with those of our relative moduli space is the Taub-NUT space itself. Thus the asymptotic form of the metric for the distinct monopoles turns out to be exact everywhere.

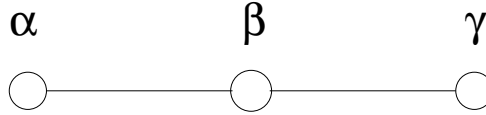
We now have the moduli space approximation of the low energy dynamics of two monopoles  $M_{\beta^*}$  and  $M_{\gamma^*}$ , and so let us come back to the electromagnetic duality in this theory, which was discussed in Sec.3. First of all we should supersymmetrize our effective action to describe the  $M_{\beta}$  and  $M_{\gamma}$  in the  $N = 4$  supersymmetric theory[20]. The ground state of such a supersymmetric Hamiltonian is given by the normalizable self-dual two-form. In the Taub-NUT space there exists unique such a two-form,  $\Omega = dQ$ , where  $Q = (1 + 2l/r)^{-1}(dq + \mathbf{w}(\mathbf{r}) \cdot d\mathbf{r})$  is the one-form associated with the conserved relative charge[23, 2]. This ground state is interpreted naturally as

the threshold bound state corresponding to the fundamental monopole  $M_{\alpha}$ , whose existence is necessary for the electromagnetic duality to hold.

The moduli space of distance monopoles in larger group has been discussed in detail in Ref.[2]. The threshold bound state of these monopoles has been found by Gibbons[27].

## 5.2 $SU(4) \rightarrow U(1)^2 \times SU(2)$

Let us now consider the case where the  $SU(4)$  gauge symmetry is partially broken to  $U(1)^2 \times SU(2)$ . In the maximally broken case, the simple roots  $\alpha, \beta, \gamma$  are chosen so that their inner products with  $\mathbf{h}$  are positive. The root diagram of  $SU(4)$  is shown in Fig.2. We take the limit where  $\mathbf{h} \cdot \beta = 0$  so that the unbroken  $SU(2)$  symmetry is associated to the  $\beta$  root.



**Figure 2:** Root diagram for  $SU(4)$ .

Let us consider the duality in this case. The supersymmetric multiplets of elementary charged and neutral particles can be listed in a four-by-four hermitian matrix as follows:

$$\begin{pmatrix} \gamma & W_{\alpha} & W_{\alpha+\gamma} \\ W_{\alpha}^* & \text{gluons} & W_{\gamma} \\ W_{\alpha+\gamma}^* & W_{\gamma}^* & \gamma' \end{pmatrix}. \quad (19)$$

The diagonal elements are made of two photons and a gluon of the unbroken  $SU(2)$  gauge group. The vector boson  $W_{\beta}$  is a part of the gluon spectrum for the unbroken  $SU(2)$  gauge group.

The charged vector bosons  $W_{\alpha}$  and  $W_{\gamma}$  belong to the spinor representation of  $SU(2)$ . There are also charged vector bosons  $W_{\alpha+\gamma}$ , which are

neutral under the  $SU(2)$  gauge group. In the magnetic sector, there are two massive fundamental monopoles  $M_{\alpha}$  and  $M_{\gamma}$ . Also there is a massless monopole  $M_{\beta}$ , which can be regarded as the dual of the massless gluon  $W_{\beta}$ . The magnetic charge sector are shown as follows:

$$\begin{pmatrix} \gamma & M_{\alpha} & ? \\ M_{\alpha}^* & \text{gluons} & M_{\gamma} \\ ? & M_{\gamma}^* & \gamma' \end{pmatrix}. \quad (20)$$

The classical monopole configuration corresponding to the question mark has the magnetic charge

$$\mathbf{g} = \frac{4\pi}{e}(\boldsymbol{\alpha} + \boldsymbol{\beta} + \boldsymbol{\gamma}), \quad (21)$$

which does not have any nonabelian component of magnetic charge. While this configuration has the right quantum number as the dual configuration for  $W_{\alpha+\gamma}$ , which is color neutral, it is a composite of two massive and one massless monopoles and so has 12 zero modes. For the duality to hold even in the case with unbroken nonabelian symmetry, there should be a unique state in the quantum mechanics of the above monopole configuration, which may be realized as a threshold bound.

Similar to the  $SU(3)$  case, we want to see this bound state, if exists, in the low energy effective Lagrangian, which is described by the moduli space metric. The metric of the moduli space in this case is obtained by taking the massless limit of the metric (14). After separating out the flat center-of-mass metric, the metric of the relative moduli space is 8 dimensional. The metric is written in terms of the relative position vector  $\mathbf{r}_1$  between the  $\boldsymbol{\alpha} - \boldsymbol{\beta}$  monopoles, the  $\mathbf{r}_2$  between the  $\boldsymbol{\beta} - \boldsymbol{\gamma}$  monopoles, and the two relative phases  $\psi_1, \psi_2$ . The resulting metric is the so-called Calabi-Taubian metric[4, 28],

$$ds^2 = C_{AB} d\mathbf{r}_A \cdot d\mathbf{r}_B + C_{AB}^{-1} (d\psi_A + \mathbf{w}(\mathbf{r}_A) \cdot d\mathbf{r}_A) (d\psi_B + \mathbf{w}(\mathbf{r}_B) \cdot d\mathbf{r}_B) \quad (22)$$

where

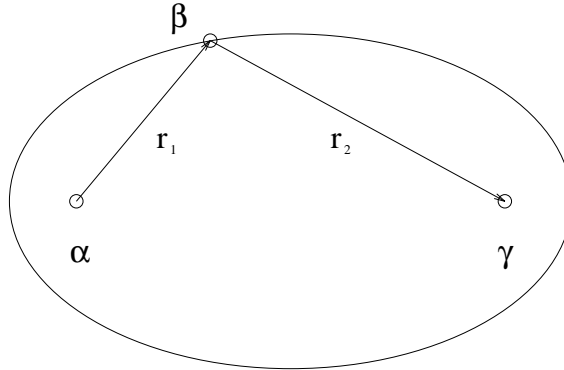
$$C_{AB} = \begin{pmatrix} \mu + 1/r_1 & \mu \\ \mu & \mu + 1/r_2 \end{pmatrix} \quad (23)$$

where  $\mu$  is related to the reduced mass of two massive monopoles and the coupling  $e$  is chosen so that the metric appears simple.

The isometry of the moduli space consists of the rotational group and the unbroken gauge group, under which the monopole kinetic energy is invariant. The isometry of the relative metric is then made of  $U(1) \times SU(2)_{gauge} \times SU(2)_{rot}$ . The 8 relative coordinates change their meaning in the massless limit:

- (1) Three of them are  $\mathbf{r}_1 + \mathbf{r}_2$ , which describes the  $SU(2)_{gauge}$  gauge invariant relative position between two massive monopoles  $\alpha$  and  $\gamma$ .
- (2) One of them is the conjugate phase of the  $SU(2)_{gauge}$  invariant relative charge  $q_1 + q_2$  of two massive monopoles.
- (3) One of them is  $a = r_1 + r_2$  which is the  $SU(2)_{gauge}$  invariant length parameter and is the total distance of line connecting  $\alpha, \beta, \gamma$  monopoles
- (4) Three of them make the three dimensional gauge orbit of  $SU(2)$ .

The structure of the monopole configuration can be learned by studying the  $SO(5)$  case, where the explicit monopole solution is known[29]. The parameter  $a$  characterizes the size of the massless nonabelian cloud around the massive monopoles. In our case, the profile of the size is given by the ellipsoid, whose focal points are two massive monopoles. Figure 3 shows such a three magnetic monopole configuration. The nonabelian magnetic charge of two massive monopoles got shielded outside the ellipsoid parameterized by  $a$ .



**Figure 3:** The  $\alpha + \beta + \gamma$  system in the case  $SU(4) \rightarrow U(1)^2 \times SU(2)$ .

Once the moduli space is known, we ask whether there exists a unique threshold bound state of the magnetic charge([4]). Once such bound is found, the electromagnetic duality can be said to hold even when the unbroken symmetry is partially nonabelian. However, there is a reason to doubt its existence. When two massive monopoles overlap each other, the relative moduli space is  $R^4$ , which does not have any the normalizable self-dual two-form. If there exists no threshold bound state, the nature of the duality has to be reexamined more carefully.

The moduli space teaches us something else too. The massive magnetic monopoles,  $M_{\alpha}, M_{\gamma}$  belong to the fundamental representation of the magnetic  $SU(2)$  group. We need to add some massless monopoles, which belongs to the adjoint representation of the unbroken simple group, to make the configuration to be a magnetic gauge singlet. The moduli space metric, more precisely, the classical monopole field configuration, tells us how massless monopoles behave around the massive monopoles. While there is no magnetic flux confinement, the massless monopole configuration looks like that of a string connecting two massive monopoles. This is a dual version of mesons, where quarks are replaced by massive monopoles and gluons are replaced by massless monopoles.

The above consideration can be extended further. Let us consider the example where  $SU(4) \rightarrow SU(3) \times U(1)$  so that  $M_{\alpha}$  is massive and  $M_{\beta}, M_{\gamma}$  are massless with the Dynkin diagram shown in Figure 2. The magnetic charge without the nonabelian magnetic component is

$$\mathbf{g} = \frac{4\pi}{e}(3\alpha + 2\beta + \gamma). \quad (24)$$

While we do not know the moduli space in this case, we can deduce a few facts about this configuration. The number of zero modes corresponding to massless monopoles is 12, among which 8 will be the dimension of the gauge orbit of  $SU(3)$ . The rest 4 will be the gauge invariant cloud shape parameters. The massive magnetic monopoles belong to the triplet of the unbroken  $SU(3)$  magnetic group. Thus this configuration is that of a proton in the dual picture, where again quarks correspond to massive monopoles and gluons to massless monopoles. If we understand how the massless monopoles are arranged, we may have a better understanding of the confining string profile inside a proton, assuming that the mass of quarks is much larger than the QCD scale and their mutual distance is larger than that of confinement.

In the case where  $SU(3) \rightarrow SU(2) \times U(1)$  with magnetic charge in Eq.(10), the magnetic moduli space has been found somewhat earlier by Dancer[30]. This moduli space of this configuration of two identical massive monopoles and one massless monopole is somewhat similar to the  $SU(4) \rightarrow SU(2) \times U(1)^2$  case. While the metric is more complicated when two massive monopoles are close to each other, we expect that the nonabelian cloud may be arranged into an ellipsoid shape when two massive monopoles get separated in large distance. This would be a dual version of the baryon in the theory with  $SU(2)$  gauge group. In the dual picture the two identical massive monopoles would be the identical quarks in the spinor representation. It may be worthy to explore further this moduli space in our context.

## 6 Concluding Remarks and Discussions

In this talk we have reviewed our recent work on the electromagnetic duality in the  $N = 4$  supersymmetric Yang-Mills theories. We have considered more general gauge group of higher rank, which is broken spontaneously to maximally or partially. To match the magnetic monopoles spectrum to that of electric charges, we need to understand the low energy dynamics of BPS magnetic monopoles by the moduli space approximation. Some magnetic monopole states, which are needed for the duality to hold, appear only quantum mechanically as threshold bounds of classical fundamental monopoles. We discuss in detail the  $SU(3) \rightarrow U(1)^2$  and  $SU(4) \rightarrow U(1)^2 \times SU(2)$  cases.

There are some closely related ideas I have not discussed here for the lack of time and space, which can be seen in the original papers. Most interestingly there are a lot detailed discussion of the case where the unbroken group has a nonabelian component. We believe the further investigation along this direction would be fruitful.

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